IIR Filter Designing Using Butterworth Approximation

1 What is Butterworth Approximation?

1. Butterworth lowpas filter (LPF) was proposed by Butterworth in 1930 in his paper titled: *On the Theory of Filter Amplifiers* (Link: `https://www.changpuak.ch/electronics/downloads/On_the_Theory_of_Filter_Amplifiers.pdf`).
2. He proposed that, any filter with its frequency response $H(\Omega)$ that satisfies the equation
$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \tag{1}$$
is a low pass filter with order N and cutoff frequency Ω_c.
3. The Laplace transform $H(s)$ follows the equation,
$$H(s)H(-s) = \frac{1}{1+\left(\frac{s}{j\Omega_c}\right)^{2N}} \tag{2}$$
4. The roots of (2) are given by,
$$s_k = \Omega_c e^{j\frac{\pi}{2}\frac{2k+1+N}{N}} \tag{3}$$
where, $k = \{0, 1, 2, \ldots, 2N-1\}$.
5. We consider the roots that lie in the left hand side of the $j\Omega$ axis which are obtained by varying k from 0 to $N-1$.
6. Therefore, the Butterworth low pass filter is given as,
$$H(s) = \frac{1}{\prod_{k=0}^{N-1}(s-s_k)} \tag{4}$$
where, s_k is given by (3).

2 Bilinear Transform

The Butterworth approximation for analog filters can be used for construction digital IIR filters using Bilinear transform. The Bilinear transform establishes a relationship between s (Laplace) and z (Z transform). The mapping, given by *Bilinear Transform*, is given by the equation
$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \tag{5}$$

Where, T is the sampling duration. Due to this transformation, a non-linear mapping from Ω to ω is obtained, defined by,

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right) \tag{6}$$

Or,

$$\omega = 2\tan^{-1}\left(\frac{\Omega T}{2}\right) \tag{7}$$

The above effect is known as *frequency warping.* In order to remove this warping effect, the analog frequencies are *prewarped* using the equation 6.

Point to be noted is that, for simplicity of calculations, we shall consider $T = 2$. Considering this won't cause any harm as the factor $T/2$ would get cancelled while doing Bilinear transformation. Hence the equation (6) to be used for prewarping reduces to,

$$\Omega = \tan\left(\frac{\omega}{2}\right) \tag{8}$$

And equation (7) reduces to,

$$\omega = 2\tan^{-1}\Omega \tag{9}$$

The normalized denominator polynomials for different order N are as shown in Table 1.

Table 1: Denominator Polynomials for Butterworth Filters with Order N

Order (N)	Denominator Polynomial
1	$s+1$
2	$s^2+1.414s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.766s+1)(s^2+1.848s+1)$
5	$(s+1)(s^2+0.618s+1)(s^2+1.618s+1)$
6	$(s^2+0.518s+1)(s^2+1.414s+1)(s^2+1.932s+1)$
7	$(s+1)(s^2+0.445s+1)(s^2+1.247s+1)(s^2+1.802s+1)$

3 Designing Digital IIR Low Pass Filter

Example: Design a digital low pass filter with specifications as:

$$-2dB \le |H(\omega)| \le 0 \qquad = 0 \le \omega \le 0.2\pi$$

$$|H(\omega)| \le -15dB \qquad = 0.5\pi \le \omega$$

- **Step 0: Interpreting the specifications and prewarping the frequencies.**

  Following specifications can be observed:
  Passband edge: $\omega_p = 0.2\pi$.
  Stopband edge: $\omega_s = 0.5\pi$.
  Passband attenuation: $A_p = -2dB$.
  Stopband attenuation: $A_s = -15dB$.

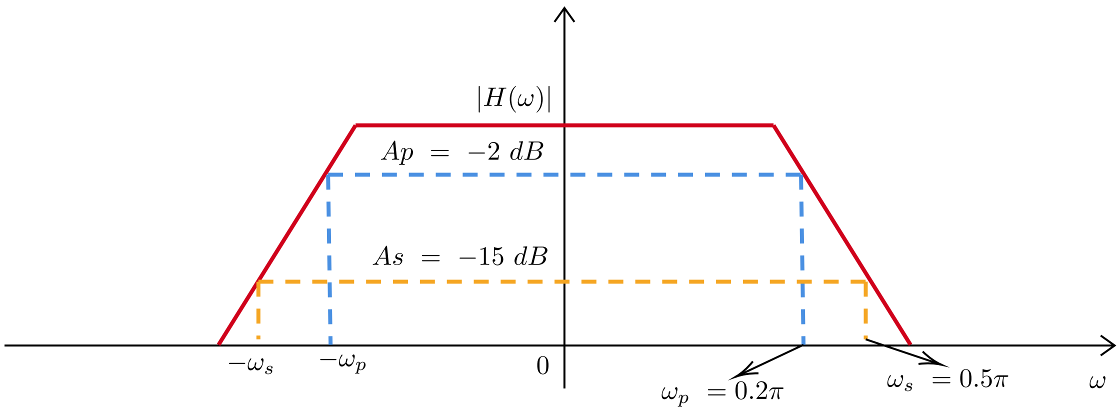

Figure 1: Low Pass Filter Specifications

Prewarping:

$$\Omega_{p,LP} = \tan\left(\frac{\omega_p}{2}\right) = \tan\left(\frac{0.2\pi}{2}\right) = 0.3249$$

$$\Omega_{s,LP} = \tan\left(\frac{\omega_s}{2}\right) = \tan\left(\frac{0.5\pi}{2}\right) = 1$$

- **Step 1: Frequency Mapping For Obtaining Normalized LPF**

  Generally, for design of any IIR filter, the pass band edge of prototype LPF is mapped to 1 and corresponding changes that occur are as,

$$\Omega_{mapped} = \frac{\Omega}{\Omega_{p,LP}} \tag{10}$$

  So, we have,

$$\Omega_p = 1$$
$$\Omega_s = \frac{\Omega_{s,LP}}{\Omega_{p,LP}} = \frac{1}{0.3249} = 3.0779$$

  The mapping is as shown in Fig. 2

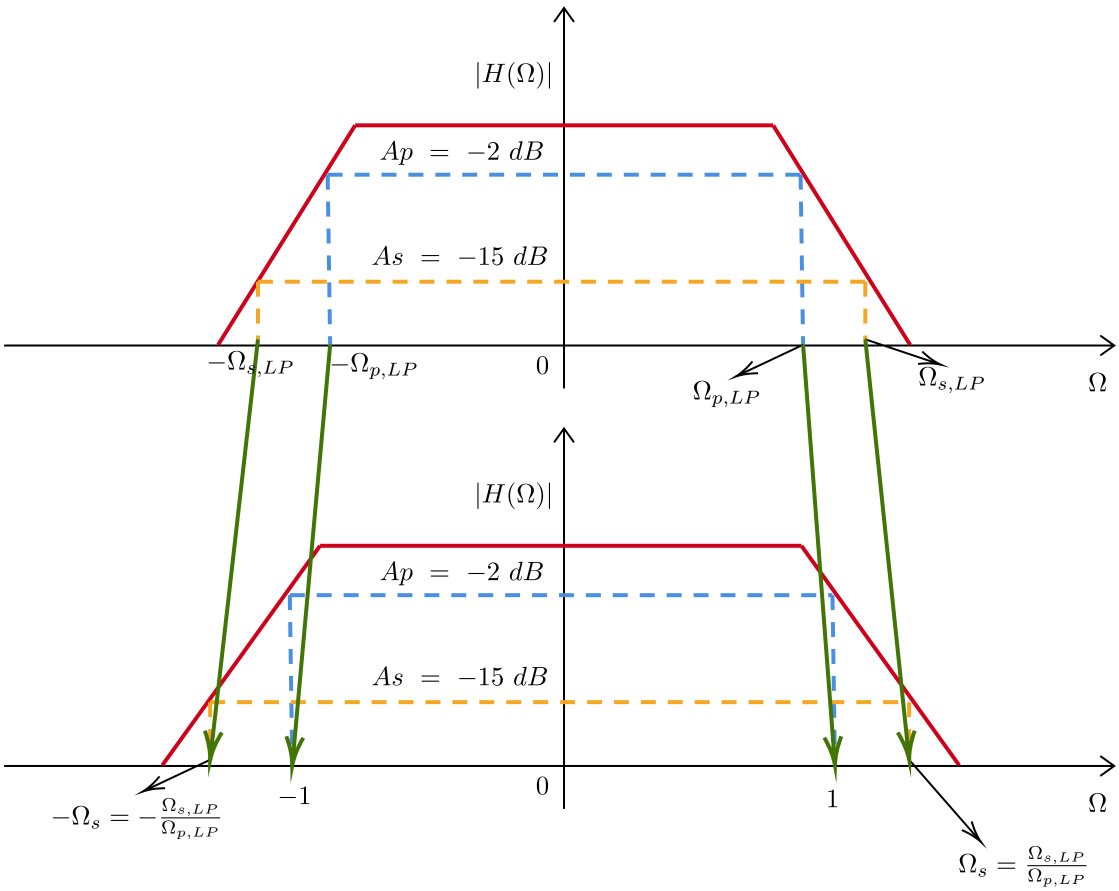

Figure 2: Mapping For Low Pass filter

- **Step 2: Finding Order for the Prototype LPF**
  In order to find the LPF, first we need to find the order. The order can be found using the formula:

$$N = \frac{\log\left(\frac{10^{|A_s|/10}-1}{10^{|A_p|/10}-1}\right)}{2 \times \log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

  For considered example,

$$N = \frac{\log\left(\frac{10^{15/10}-1}{10^{2/10}-1}\right)}{2 \times \log\left(\frac{3.0779}{1}\right)} = \lceil 1.76 \rceil = 2.$$

- **Step 3: Find Normalized LPF Equation by Finding roots or Using Polynomial Table**

  We have,

$$s_k = e^{j\frac{\pi}{4}(2k+3)} \tag{11}$$

  As $N = 2$, $k = \{0, 1\}$. So, we have,

$$s_o = e^{j\frac{3\pi}{4}}$$

and

$$s_1 = e^{j\frac{5\pi}{4}}$$

.

Using the roots, the value of $H(s)$ can be found as,

$$\begin{aligned} H(s) &= \frac{1}{(s-s_0)(s-s_1)} \\ &= \frac{1}{(s-e^{j\frac{3\pi}{4}})(s-e^{j\frac{5\pi}{4}})} \\ H(s) &= \frac{1}{s^2+1.414s+1} \end{aligned}$$

- **Step 4: Find the cutoff frequency.**

  The cutoff frequency of the filter is unknown and can be found as,

  $$\Omega_c = \frac{\Omega_s}{\left(10^{|A_s|/10}-1\right)^{\frac{1}{2N}}} \tag{12}$$

  For, this example,

  $$\Omega_c = \frac{3.0779}{\left(10^{15/10}-1\right)^{\frac{1}{4}}} = 1.3084 \tag{13}$$

- **Step 5: Find Denormalized Cutoff Frequency**

  This cutoff frequency is not the actual cutoff frequency but it is frequency when passband edge is mapped to 1. So, the actual cutoff is given by,

  $$\Omega_{c,actual} = \Omega_c \times \Omega_{p,LP} = 1.3084 \times 0.3249 = 0.4251 \tag{14}$$

- **Step 6: Find Analog Filter**

  In order to find LPF, substitute $s \to \frac{s}{\Omega_{c,actual}}$.

  $$\begin{aligned} H(s)|_{s\to\frac{s}{\Omega_{c,actual}}} &= \frac{1}{\left(\frac{s}{\Omega_{c,actual}}\right)^2 + 1.414\left(\frac{s}{\Omega_{c,actual}}\right)+1} \\ &= \frac{\Omega_{c,actual}^2}{s^2+1.414\Omega_{c,actual}s+\Omega_{c,actual}^2} \\ &= \frac{0.4251^2}{s^2+1.414\times 0.4251s+0.4251^2} \\ &= \frac{0.181}{s^2+0.601s+0.181} \end{aligned}$$

- **Step 7: Find the equivalent IIR filter using Bilinear Transform**

Substitute $s \to \frac{1-z^{-1}}{1+z^{-1}}$.

$$\begin{aligned} H(z) &= \frac{0.181}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.601\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.181} \\ &= \frac{0.181(1+2z^{-1}+z^{-2})}{1.782 - 1.638z^{-1} + 0.58z^{-2}} \\ H(z) &= \frac{0.102 + 0.204z^{-1} + 0.102z^{-2}}{1 - 0.919z^{-1} + 0.325z^{-2}} \end{aligned}$$

Final Digital Low pass IIR filter is:

$$H(z) = \frac{0.102 + 0.204z^{-1} + 0.102z^{-2}}{1 - 0.919z^{-1} + 0.325z^{-2}} \tag{15}$$

MATLAB Code

Following is the code snippet. The frequency response and the pole zero plot is as shown in Fig. 3 and 4.

```
clc;clear;close all;
%% Pass band and Stop Band edges
wp = 0.2;
ws = 0.5;

%% Pass band and Stop Band attenuation
Ap = 2;
As = 15;

%% Get Cutoff and Order
[N wc] = buttord(wp,ws,Ap,As);
disp(['Order:' num2str(N)]);
disp(['Cutoff: ' num2str(wc)]);

%% Value of Cutoff in Analog equivalent
WW = tan(wc.*pi/2);
disp(['Cutoff in Analog: ' num2str(WW)]);

%% Get Filter Coefficients
[b a] = butter(N,wc)
%% Get Pole-Zero Plot
figure;
zplane(b,a);
set(findall(gcf,'Type','line'),'LineWidth',2,'MarkerSize',40)

%% Get Frequency Response
figure;
freqz(b,a);
set(findall(gcf,'Type','line'),'LineWidth',2,'MarkerSize',40)
```

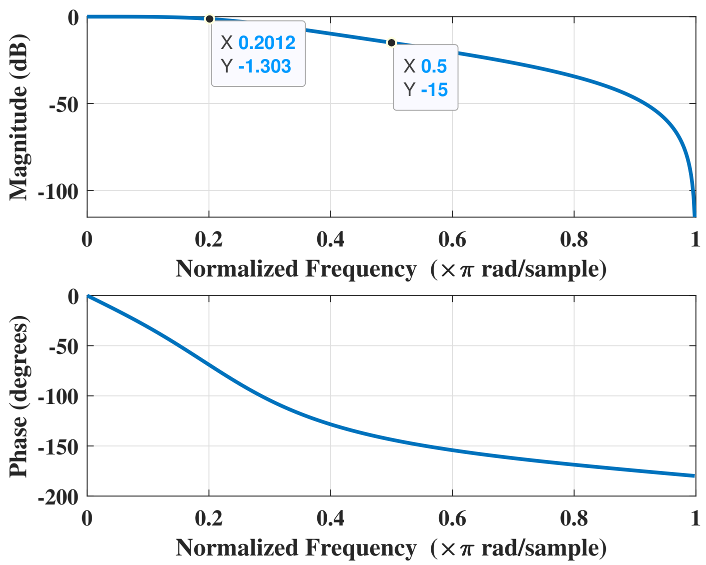

Figure 3: Frequency response for LPF

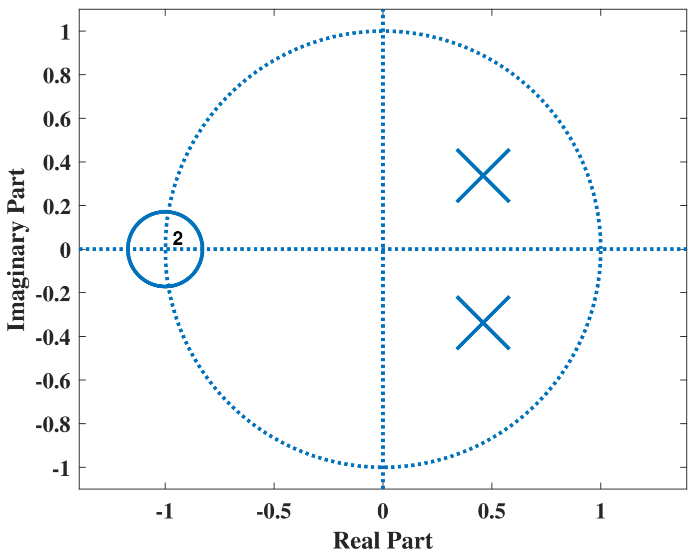

Figure 4: Pole zero for LPF

4 Designing Digital IIR High Pass Filter

Example: Design a high pass filter with following specifications:

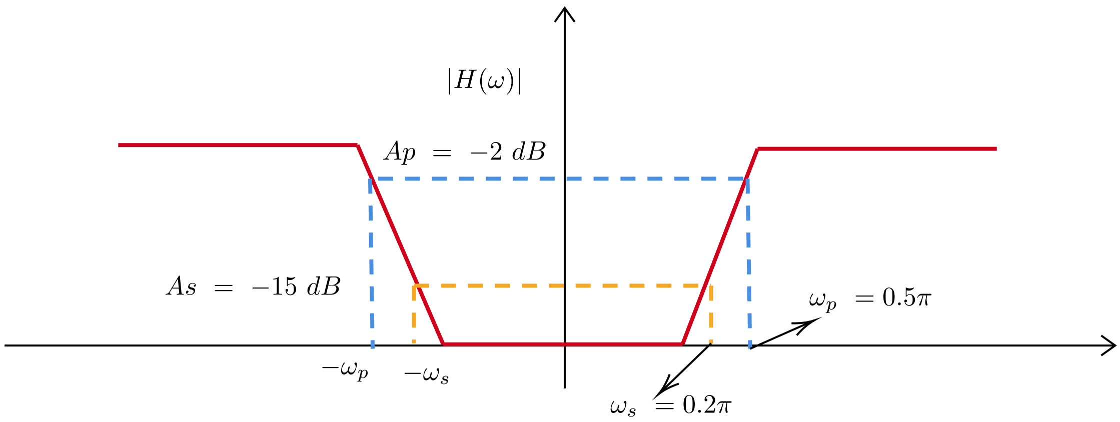

Figure 5: High pass filter specification

- **Step 0: Interpreting the specifications and prewarping the frequencies.**

  Following specifications can be observed:
  Passband edge: $\omega_p = 0.5\pi$.
  Stopband edge: $\omega_s = 0.2\pi$.
  Passband attenuation: $A_p = -2dB$.
  Stopband attenuation: $A_s = -15dB$.

  Prewarping of frequencies:

$$\Omega_{p,HP} = \tan\left(\frac{\omega_p}{2}\right) = \tan\left(\frac{0.5\pi}{2}\right) = 1$$

$$\Omega_{s,HP} = \tan\left(\frac{\omega_s}{2}\right) = \tan\left(\frac{0.2\pi}{2}\right) = 0.3249$$

- **Step 1: Frequency Mapping for Obtaining Normalized LPF**

  Butterworth approximation is for Lowpass filter only. If a high pass filter is to be constructed, then it should be mapped to the low pass filter. A typical mapping for a high pass filter is done as,

$$s \to \frac{1}{s} \tag{16}$$

  In frequency terms, it is obtained as,

$$\Omega_{LP} = -\frac{1}{\Omega_{HP}} \tag{17}$$

  It is a typical practice that the pass band edge of the lowpass filter is mapped to 1. For that, the RHS of above equation us scaled by Ω_p of the highpass. So, we have,

$$\Omega_{LP} = -\frac{\Omega_{p,HP}}{\Omega_{HP}} \tag{18}$$

  The frequency mapping is as shown in Fig. 6.

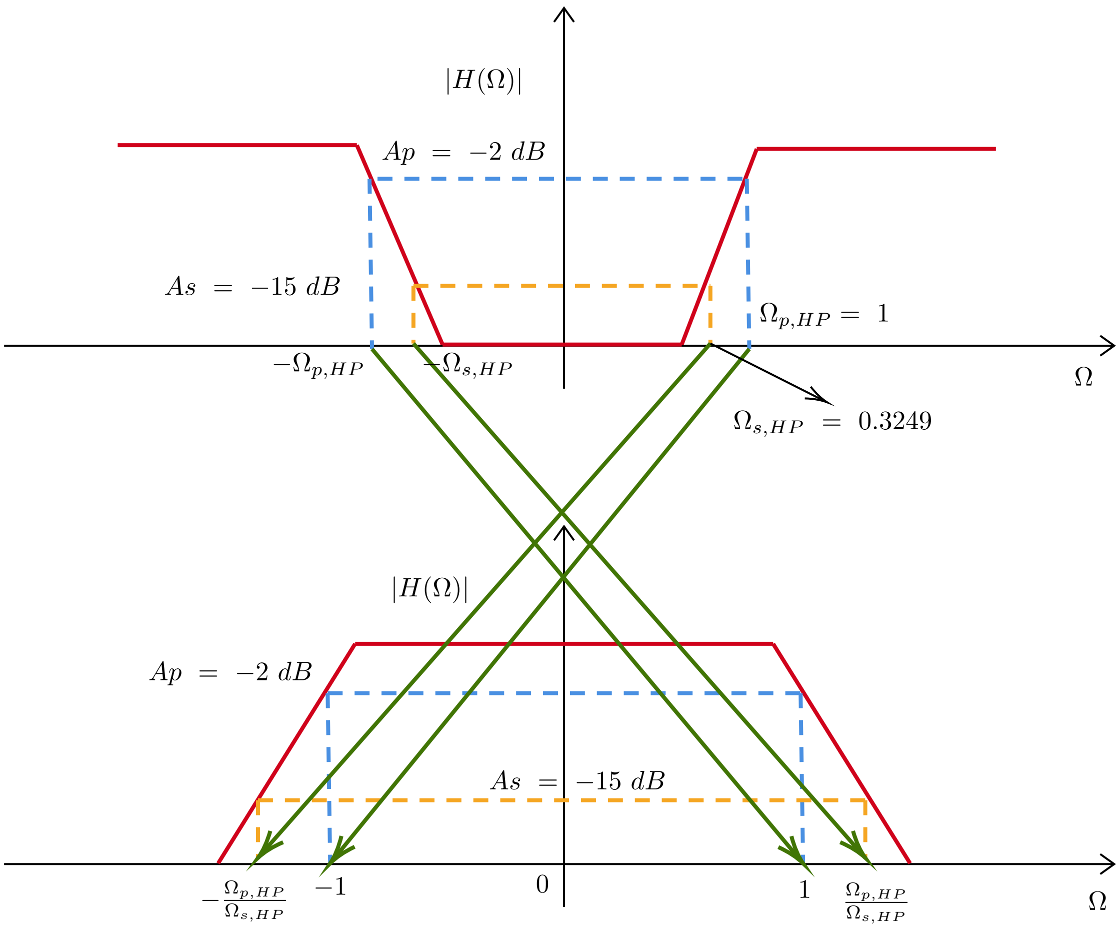

Figure 6: Frequency mapping for HPF

It can be observed that $-\Omega_p$ is mapped to 1 of LPF and $-\Omega_s$ is mapped to $\frac{\Omega_p}{\Omega_s}$. Therefore, the mapped passband and stop band edge of the LPF with passband assumed at 1 is

$$\Omega_p = 1$$

$$\Omega_s = \frac{\Omega_{p,HP}}{\Omega_{s,HP}} = \frac{1}{0.3249} = 3.078$$

- **Step 2: Finding Order for the Prototype LPF**

  The order can be found using the formula

$$N = \frac{\log\left(\frac{10^{|A_s|/10}-1}{10^{|A_p|/10}-1}\right)}{2 \times \log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

  For considered example,

$$N = \frac{\log\left(\frac{10^{15/10}-1}{10^{2/10}-1}\right)}{2 \times \log\left(\frac{0.3249}{1}\right)} = \lceil 1.76 \rceil = 2.$$

- **Step 3: Find Normalized LPF Equation by Finding roots or Using Polynomial Table**

  We have,

$$s_k = e^{j\frac{\pi}{4}(2k+3)} \tag{19}$$

  As $N = 2$, $k = \{0, 1\}$. So, we have,

$$s_o = e^{j\frac{3\pi}{4}}$$

  and

$$s_1 = e^{j\frac{5\pi}{4}}$$

  Using the roots, the value of $H(s)$ can be found as,

$$\begin{aligned} H(s) &= \frac{1}{(s - s_0)(s - s_1)} \\ &= \frac{1}{(s - e^{j\frac{3\pi}{4}})(s - e^{j\frac{5\pi}{4}})} \\ H(s) &= \frac{1}{s^2 + 1.414s + 1} \end{aligned}$$

- **Step 4: Find the cutoff frequency.**

  It can be seen that, the mapped prototype LPF cannot have the cutoff frequency = 1. Hence, there is a need to obtain the required cutoff frequency of the LPF using the formula,

$$\Omega_c = \frac{\Omega_s}{\left(10^{|A_s|/10} - 1\right)^{\frac{1}{2N}}} \tag{20}$$

  For, this example,

$$\Omega_c = \frac{3.078}{\left(10^{15/10} - 1\right)^{\frac{1}{4}}} = 1.3085 \tag{21}$$

- **Step 5: Find Denormalized Cutoff Frequency**

  This cutoff frequency is for the LPF mapped with passband edge to 1. The actual cutoff frequency can be calculated as,

$$\Omega_{c,actual} = \frac{\Omega_{p,HP}}{\Omega_c} = \frac{1}{3.078} = 0.764 \tag{22}$$

- **Step 6: Find the Analog Filter**

  For mapping the LPF to HPF we need to substitute,

$$s \to \frac{\Omega_{c,actual}}{s} \tag{23}$$

  to get the required HPF. Therefore, the HPF obtained is as shown below.

$$\begin{aligned} H(s) &= \frac{1}{\frac{\Omega_{c,actual}^2}{s^2} + 1.414\frac{\Omega_{c,actual}}{s} + 1} \\ &= \frac{s^2}{s^2 + 1.414\Omega_{c,actual}s + \Omega_{c,actual}^2} \\ &= \frac{s^2}{s^2 + 1.081s + 0.584} \end{aligned}$$

- **Step 7: Find the equivalent IIR filter using Bilinear Transform.**

  We have $s \rightarrow \frac{1-z^{-1}}{1+z^{-1}}$. Therefore,

$$\begin{aligned}H(z) &= \frac{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 1.081\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.584} \\ &= \frac{1-2z^{-1}+z^{-2}}{2.6657-0.831z^{-1}+0.5035z^{-2}} \\ &= \frac{0.375-0.751z^{-1}+0.375z^{-2}}{1-0.3126z^{-1}+0.1891z^{-2}}\end{aligned}$$

  The final High pass filter is obtained as,

$$H(z) = \frac{0.375-0.751z^{-1}+0.375z^{-2}}{1-0.3126z^{-1}+0.1891z^{-2}} \tag{24}$$

MATLAB Code

Following is the code snippet. The frequency response and the pole zero plots are as shown in Fig. 7 and 8 respectively.

```
clc;clear;close all;
%% Pass band and Stop Band edges
wp = 0.5;
ws = 0.2;

%% Pass band and Stop Band attenuation
Ap = 2;
As = 15;

%% Get Cutoff and Order
[N wc] = buttord(wp,ws,Ap,As);
disp(['Order:' num2str(N)]);
disp(['Cutoff: ' num2str(wc)]);

%% Value of Cutoff in Analog equivalent
WW = tan(wc.*pi/2);
disp(['Cutoff in Analog: ' num2str(WW)]);

%% Get Filter Coefficients
[b a] = butter(N,wc,'high')
%% Get Pole-Zero Plot
figure;
zplane(b,a);
set(findall(gcf,'Type','line'),'LineWidth',2,'MarkerSize',40)

%% Get Frequency Response
figure;
freqz(b,a);
set(findall(gcf,'Type','line'),'LineWidth',2,'MarkerSize',40)
```

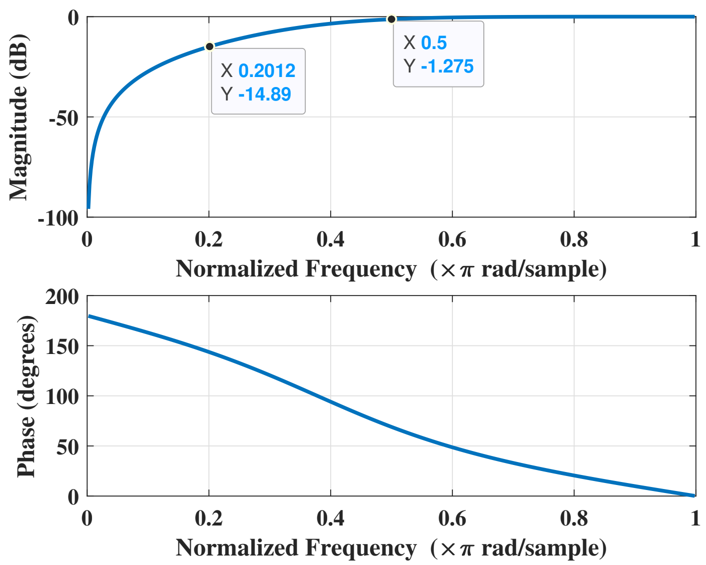

Figure 7: Frequency response for HPF

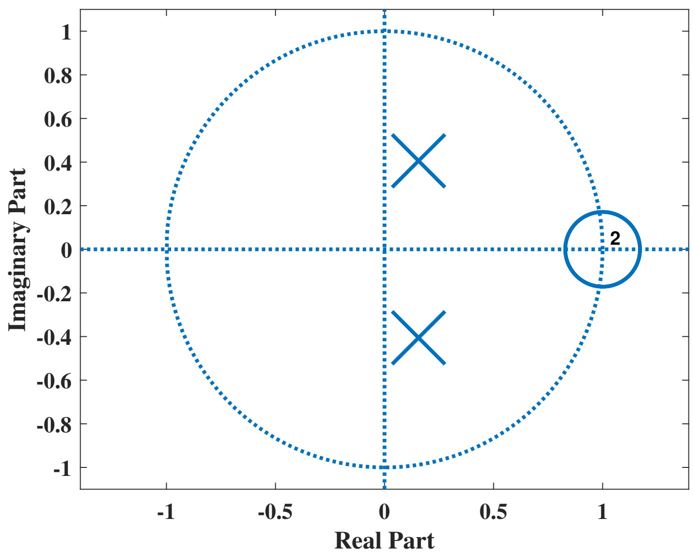

Figure 8: Pole zero for HPF

5 Designing Digital IIR Band Pass Filter

Example: Design the Digital IIR Bandpass filter with following secifications:

1. Lower stop band edge: 0.1π

2. Lower pass band edge: 0.4π
3. Higher pass band edge: 0.6π
4. Higher stop band edge: 0.9π
5. Pass band attenuation: -3 dB
6. Stop band attenuation: -18 dB

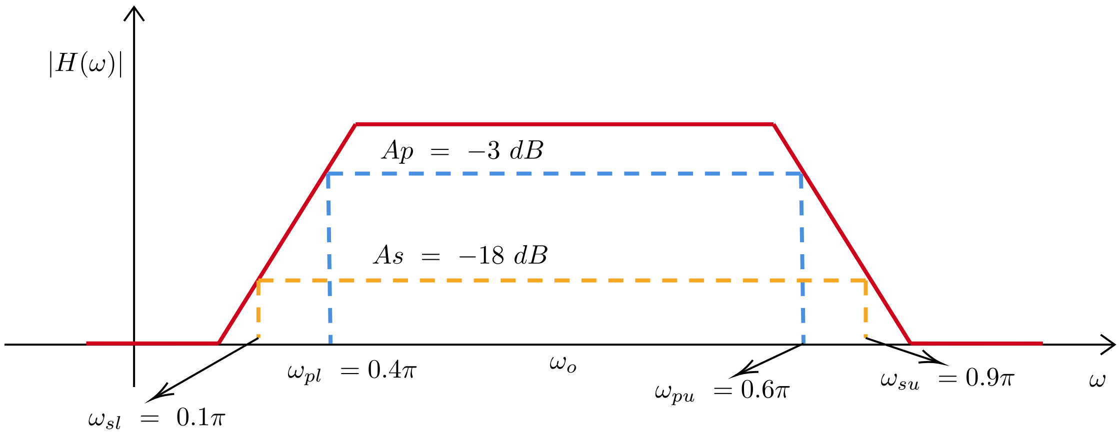

Figure 9: Band Pass Filter Specification

- **Step 0: Interpreting the specifications and prewarping the frequencies**

$$\Omega_{sl} = \tan\left(\frac{\omega_{sl}}{2}\right) = \tan\left(\frac{0.1\pi}{2}\right) = \tan(0.05\pi) = 0.15838$$

$$\Omega_{pl} = \tan\left(\frac{\omega_{pl}}{2}\right) = \tan\left(\frac{0.4\pi}{2}\right) = \tan(0.2\pi) = 0.72654$$

$$\Omega_{pu} = \tan\left(\frac{\omega_{pu}}{2}\right) = \tan\left(\frac{0.6\pi}{2}\right) = \tan(0.3\pi) = 1.37638$$

$$\Omega_{su} = \tan\left(\frac{\omega_{su}}{2}\right) = \tan\left(\frac{0.9\pi}{2}\right) = \tan(0.45\pi) = 6.31375$$

- **Step 1: Frequency Mapping for Obtaining Normalized LPF**

The bandpass filter is to be mapped to low pass filter in order to use Butterworth approximation. The mapping used is:

$$s \to \frac{s^2 + \Omega_o^2}{Bs} \tag{25}$$

where, $B = \Omega_{pu} - \Omega_{pl}$ and Ω_o is the center frequency which is geometric mean of any two frequencies on its either side that correspond to same magnitude. In general, the passband frequencies are considered. Therefore, $\Omega_o^2 = \Omega_{pu} \times \Omega_{pl}$. The mapping reduces to,

$$s \to \frac{s^2 + \Omega_{pu} \times \Omega_{pl}}{s(\Omega_{pu} - \Omega_{pl})} \tag{26}$$

and the frequency mappings are given as,

$$\Omega_{LP} \rightarrow \frac{\Omega_{BP}^2 - \Omega_{pu} \times \Omega_{pl}}{\Omega_{BP}(\Omega_{pu} - \Omega_{pl})} \tag{27}$$

where, Ω_{BP} and Ω_{LP} are the frequencies corresponding to BPF and prototype LPF respectively. We have,

$$B = \Omega_{pu} - \Omega_{pl} = 0.64984$$
$$\Omega_o^2 = \Omega_{pu} \times \Omega_{pl} = 1$$

The pass band and stop band edges when mapped lead to,

$$\Omega_{sl} = 0.15838 \rightarrow \frac{0.15838^2 - 1}{0.15838 \times 0.64984} = -9.4721$$
$$\Omega_{pl} = 0.72654 \rightarrow \frac{0.72654^2 - 1}{0.72654 \times 0.64984} = -1$$
$$\Omega_{pu} = 1.37638 \rightarrow \frac{1.37638^2 - 1}{1.37638 \times 0.64984} = 1$$
$$\Omega_{su} = 6.31375 \rightarrow \frac{6.31375^2 - 1}{6.31375 \times 0.64984} = 9.4721$$
$$\Omega_0 = 1 \rightarrow \frac{1 - 1}{1 \times 0.64984} = 0$$

The mapping can be easily visualized graphically as shown in Fig. 10.

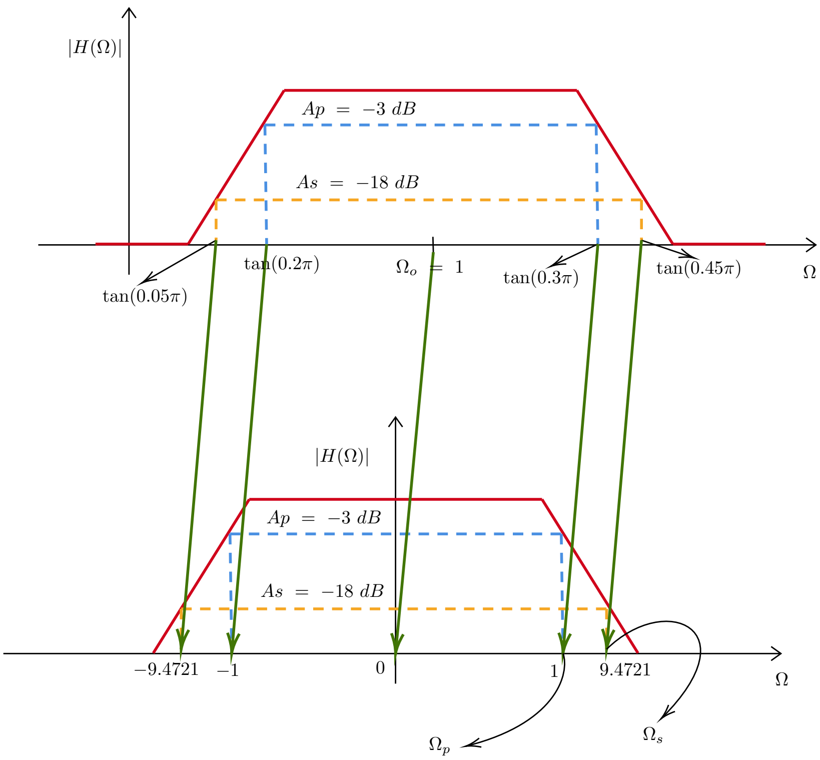

Figure 10: Mapping for Bandpass Filter

Therefore, the mapped frequencies for LPF can be obtained as,

$$\Omega_p = 1$$
$$\Omega_s = \frac{\Omega_{su}^2 - \Omega_{pu} \times \Omega_{pl}}{\Omega_{su}(\Omega_{pu} - \Omega_{pl})} = 9.4721$$

- **Step 2: Finding Order for the Prototype LPF**

  The order can be found as,

$$N = \frac{\log\left(\frac{10^{|A_s|/10}-1}{10^{|A_p|/10}-1}\right)}{2 \times \log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

  For considered example,

$$N = \frac{\log\left(\frac{10^{18/10}-1}{10^{3/10}-1}\right)}{2 \times \log\left(\frac{9.4721}{1}\right)} = \lceil 0.9192 \rceil = 1.$$

- **Step 3: Find Normalized LPF Equation by Finding roots or Using Polynomial Table**

  From Table 1, we have

$$H(s) = \frac{1}{s+1} \tag{28}$$

- **Step 4: Find the cutoff frequency.**

  The cutoff frequency can be calculated as,

$$\Omega_c = \frac{\Omega_s}{\left(10^{|A_s|/10} - 1\right)^{\frac{1}{2N}}} \tag{29}$$

  For this example,

$$\Omega_c = \frac{9.4721}{\left(10^{18/10} - 1\right)^{\frac{1}{2}}} = 1.202 \tag{30}$$

- **Step 5: Find Denormalized Cutoff Frequency**

  The cutoff frequencies for the BPF can be found as,

$$\begin{aligned}
\Omega_{c1} &= -\frac{\Omega_c B}{2} + \frac{1}{2}\sqrt{\Omega_c^2 B^2 + 4\Omega_o^2} \\
&= -\frac{1.202 \times 0.64984}{2} + \frac{1}{2}\sqrt{1.202^2 \times 0.64984^2 + 4 \times 1} = 0.6830 \\
\Omega_{c2} &= +\frac{\Omega_c B}{2} + \frac{1}{2}\sqrt{\Omega_c^2 B^2 + 4\Omega_o^2} \\
&= \frac{1.202 \times 0.64984}{2} + \frac{1}{2}\sqrt{1.202^2 \times 0.64984^2 + 4 \times 1} = 1.4641
\end{aligned}$$

- **Step 6: Find the Analog Filter**

In order to obtain the bandpass filter from lowpass filter, we substitute,

$$s \rightarrow \frac{s^2 + \Omega_o^2}{(\Omega_{c2} - \Omega_{c1})s} \tag{31}$$

Therefore, the Bandpass filter obtained after transformation,

$$\begin{aligned}
H(s) &= \frac{1}{\frac{s^2+1}{(1.4641-0.6830)s} + 1} \\
&= \frac{1}{\frac{s^2+1}{0.7811s} + 1} \\
H(s) &= \frac{0.7811s}{s^2 + 0.7811s + 1}
\end{aligned}$$

- **Step 7: Find the equivalent IIR filter using Bilinear Transform.**

  Bilinear transformation is given as $s \rightarrow \frac{1-z^{-1}}{1+z^{-1}}$. Therefore,

$$\begin{aligned}
H(z) &= \frac{0.7811\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.7811\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 1} \\
H(z) &= \frac{0.2809 - 0.2809z^{-2}}{1 + 0.4383z^{-2}}
\end{aligned}$$

  The final bandpass filter is obtained as,

$$H(z) = \frac{0.2809 - 0.2809z^{-2}}{1 + 0.4383z^{-2}} \tag{32}$$

MATLAB Code

Following is the code snippet for Bandpass filter. The frequency response and the pole zero plot is as shown in Fig. 11 and 12 respectively.

```
clc;clear;close all;
%% Pass band and Stop Band edges
wp = [0.4 0.6];
ws = [0.1 0.9];

%% Pass band and Stop Band attenuation
Ap = 3;
As = 18;

%% Get Cutoff and Order
[N wc] = buttord(wp,ws,Ap,As);
disp(['Order:' num2str(N)]);
disp(['Cutoff: ' num2str(wc)]);

%% Value of Cutoff in Analog equivalent
WW = tan(wc.*pi/2);
disp(['Cutoff in Analog: ' num2str(WW)]);

%% Get Filter Coefficients
```

```
[b a] = butter(N,wc,'bandpass')
%% Get Pole-Zero Plot
figure;
zplane(b,a);
set(findall(gcf,'Type','line'),'LineWidth',2,'MarkerSize',40)

%% Get Frequency Response
figure;
freqz(b,a);
set(findall(gcf,'Type','line'),'LineWidth',2,'MarkerSize',40)
```

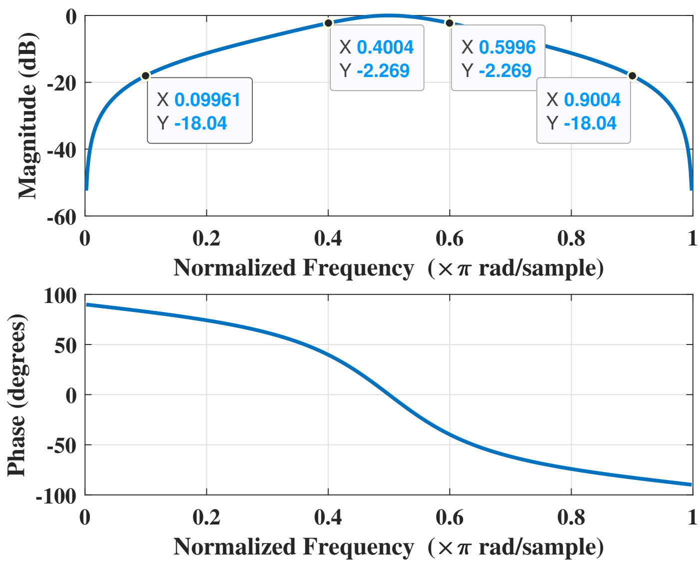

Figure 11: Frequency response for BPF

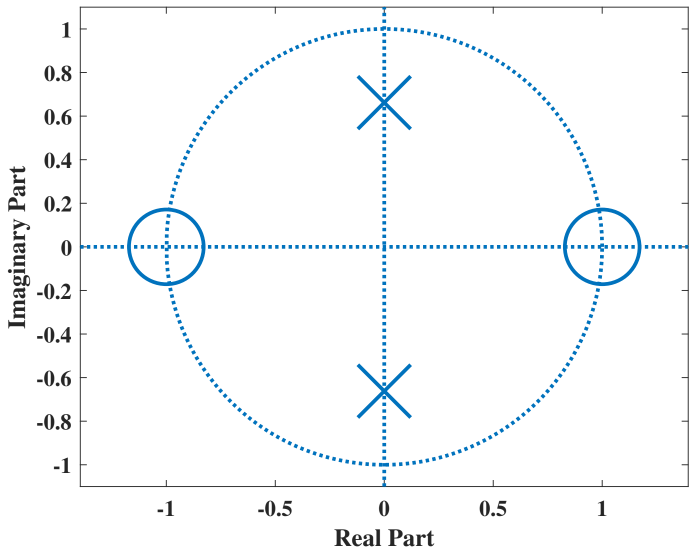

Figure 12: Pole zero for BPF

6 Designing Digital IIR Band Stop Filter

Example: Design a Digital IIR Band stop filter with following specifications.

1. Lower pass band edge: 0.1π
2. Lower stop band edge: 0.4π
3. Higher stop band edge: 0.6π
4. Higher pass band edge: 0.9π
5. Pass band attenuation: -3 dB
6. Stop band attenuation: -18 dB

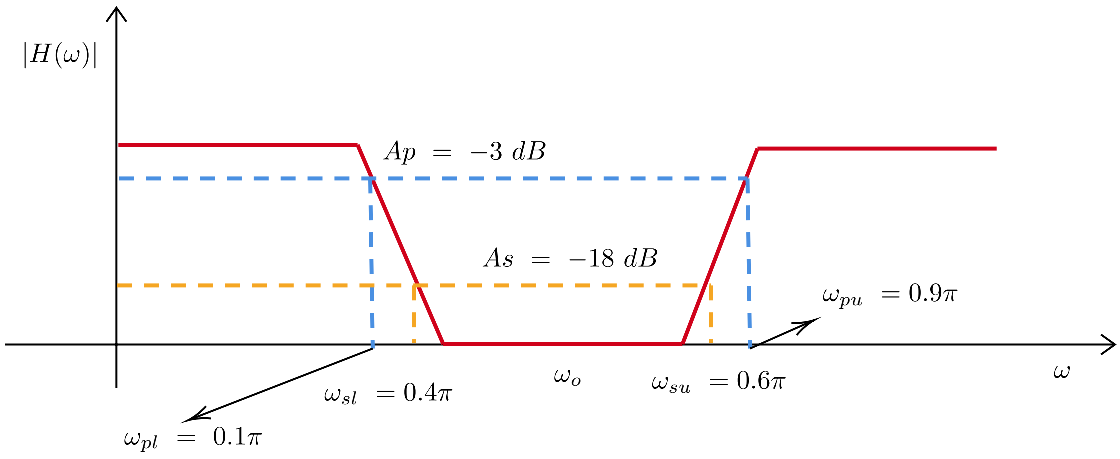

Figure 13: Band Stop Filter Specifications

- **Step 0: Interpreting the specifications and prewarping the frequencies**

$$\Omega_{pl} = \tan\left(\frac{\omega_{pl}}{2}\right) = \tan\left(\frac{0.1\pi}{2}\right) = \tan(0.05\pi) = 0.15838$$
$$\Omega_{sl} = \tan\left(\frac{\omega_{sl}}{2}\right) = \tan\left(\frac{0.4\pi}{2}\right) = \tan(0.2\pi) = 0.72654$$
$$\Omega_{su} = \tan\left(\frac{\omega_{su}}{2}\right) = \tan\left(\frac{0.6\pi}{2}\right) = \tan(0.3\pi) = 1.37638$$
$$\Omega_{pu} = \tan\left(\frac{\omega_{pu}}{2}\right) = \tan\left(\frac{0.9\pi}{2}\right) = \tan(0.45\pi) = 6.31375$$

- **Step 1: Frequency Mapping for Obtaining Normalized LPF**

  The band stop filter is to be mapped to low pass filter in order to use Butterworth approximation. The mapping used is:

$$s \to \frac{Bs}{s^2 + \Omega_o^2} \tag{33}$$

  where, $B = \Omega_{pu} - \Omega_{pl}$ and Ω_o is the center frequency which is geometric mean of any two frequencies on its either side that correspond to same magnitude. In general, the passband frequencies are considered. Therefore, $\Omega_o^2 = \Omega_{pu} \times \Omega_{pl}$. The mapping reduces to,

$$s \to \frac{s(\Omega_{pu} - \Omega_{pl})}{s^2 + \Omega_{pu} \times \Omega_{pl}} \tag{34}$$

  and the frequency mappings are given as,

$$\Omega_{LP} \to \frac{\Omega_{BS}(\Omega_{pu} - \Omega_{pl})}{\Omega_{pu} \times \Omega_{pl} - \Omega_{BS}^2} \tag{35}$$

  where, Ω_{BS} and Ω_{LP} are the frequencies corresponding to BSF and prototype LPF respectively. Here,

$$B = \Omega_{pu} - \Omega_{pl} = 6.15537$$

$$\Omega_o^2 = \Omega_{pu} \times \Omega_{pl} = 1$$

The pass band and stop band edges when mapped lead to,

$$\Omega_{pl} = 0.15838 \rightarrow \frac{6.15537 \times 0.15838}{1 - 0.15838^2} = 1$$
$$\Omega_{sl} = 0.72654 \rightarrow \frac{6.15537 \times 0.72654}{1 - 0.72654^2} = 9.4721$$
$$\Omega_{su} = 1.37638 \rightarrow \frac{6.15537 \times 1.37638}{1 - 1.37638^2} = -9.4721$$
$$\Omega_{pl} = 6.31375 \rightarrow \frac{6.15537 \times 6.31375}{1 - 6.31375^2} = -1$$
$$\Omega_o = 1 \rightarrow \frac{6.15537 \times 1}{1 - 1^2} = \infty$$

The mapping for the BSF is as shown in Fig. 14.

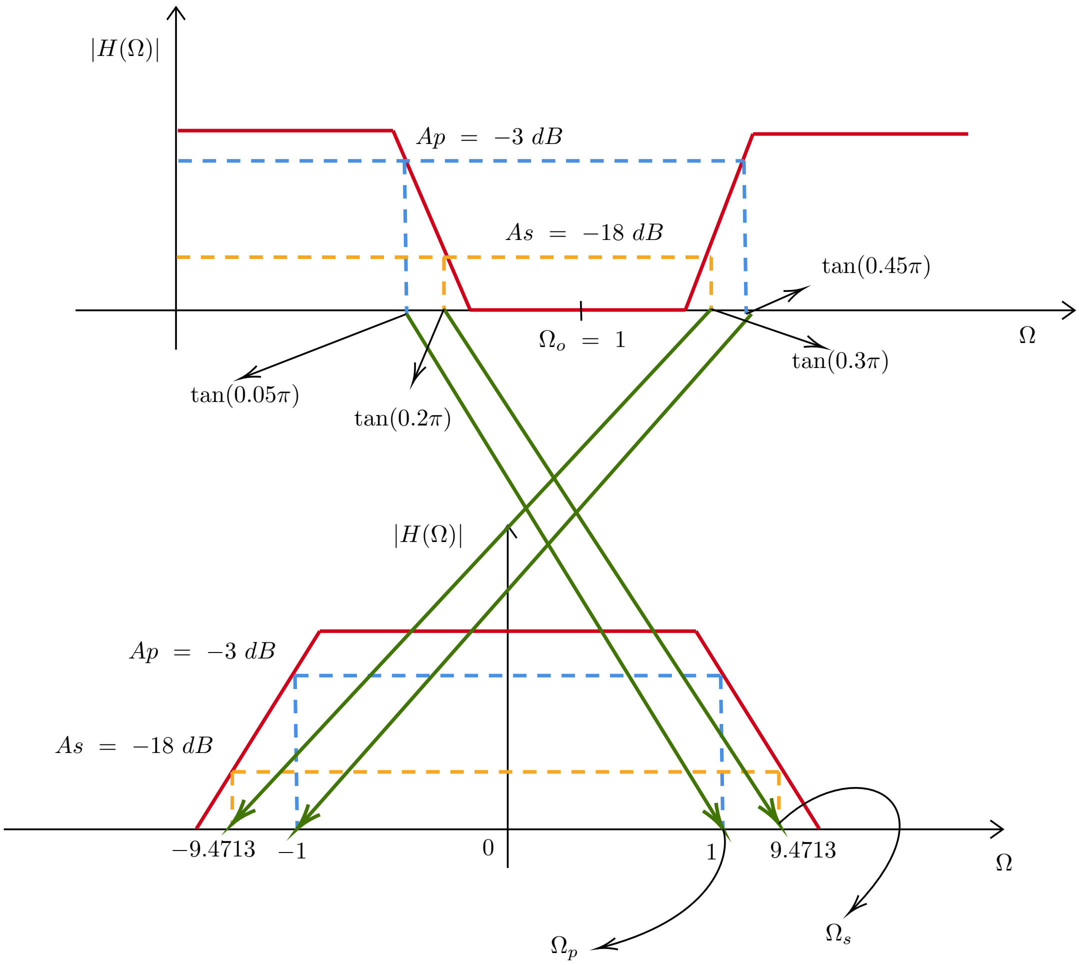

Figure 14: Frequency Mapping for Band Stop Filter

Therefore, the mapped frequencies for LPF can be obtained as,

$$\Omega_p = 1$$
$$\Omega_s = \frac{\Omega_{sl}(\Omega_{pu} - \Omega_{pl})}{\Omega_{pu} \times \Omega_{pl} - \Omega_{sl}^2} = 9.4721$$

- **Step 2: Finding Order for the prototype LPF**

  The order can be found as,

$$N = \frac{\log\left(\frac{10^{|A_s|/10}-1}{10^{|A_p|/10}-1}\right)}{2 \times \log\left(\frac{\Omega_s}{\Omega_p}\right)}$$

  For considered example,

$$N = \frac{\log\left(\frac{10^{18/10}-1}{10^{3/10}-1}\right)}{2 \times \log\left(\frac{9.4721}{1}\right)} = \lceil 0.9192 \rceil = 1.$$

- **Step 3: Find Normalized LPF Equation by Finding roots or Using Polynomial Table.**

  From Table 1, the normalized LPF is given as,

$$H(s) = \frac{1}{s+1} \tag{36}$$

- **Step 4: Find the cutoff frequency**

  The cutoff frequency can be calculated as,

$$\Omega_c = \frac{\Omega_s}{\left(10^{|A_s|/10} - 1\right)^{\frac{1}{2N}}} \tag{37}$$

  For this example,

$$\Omega_c = \frac{9.4721}{\left(10^{18/10} - 1\right)^{\frac{1}{2}}} = 1.202 \tag{38}$$

- **Step 5: Find Denormalized Cutoff Frequency**

  The cutoff frequency for the BSF can be found as,

$$\begin{aligned}
\Omega_{c1} &= -\frac{B}{2\Omega_c} + \frac{1}{2}\sqrt{\frac{B^2}{\Omega_c^2} + 4\Omega_o^2} \\
&= -\frac{6.15537}{2 \times 1.202} + \frac{1}{2}\sqrt{\frac{6.15537^2}{1.202^2} + 4 \times 1} = 0.1883 \\
\Omega_{c2} &= +\frac{B}{2\Omega_c} + \frac{1}{2}\sqrt{\frac{B^2}{\Omega_c^2} + 4\Omega_o^2} \\
&= -\frac{6.15537}{2 \times 1.202} + \frac{1}{2}\sqrt{\frac{6.15537^2}{1.202^2} + 4 \times 1} = 5.3093
\end{aligned}$$

- **Step 6: Find the Analog Filter**

  In order to obtain the bandstop filter from lowpass filter, we substitute,

$$s \rightarrow \frac{(\Omega_{c2} - \Omega_{c1})s}{s^2 + 1} \tag{39}$$

  Therefore, the Bandstop filter obtained after transformation,

$$\begin{aligned} H(s) &= \frac{1}{\frac{(5.3093 - 0.1883)s}{s^2+1} + 1} \\ &= \frac{1}{\frac{5.121s}{s^2+1} + 1} \\ H(s) &= \frac{s^2 + 1}{s^2 + 5.121s + 1} \end{aligned}$$

- **Step 7: Find the equivalent IIR filter using Bilinear Transform**

  Bilinear transformation is given as $s \rightarrow \frac{1-z^{-1}}{1+z^{-1}}$. Therefore,

$$\begin{aligned} H(z) &= \frac{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 1}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 5.121\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 1} \\ &= \frac{2 + 2z^{-2}}{7.1209 - 3.121z^{-2}} \\ H(z) &= \frac{0.2809 + 0.2809z^{-2}}{1 - 0.4383z^{-2}} \end{aligned}$$

  The final bandstop filter is obtained as,

$$H(z) = \frac{0.2809 + 0.2809z^{-2}}{1 - 0.4383z^{-2}} \tag{40}$$

MATLAB Code

Following is the code snippet. The frequency response and the pole zero plot is as shown in Fig. 15 and 16 respectively.

```
clc;clear;close all;
%% Pass band and Stop Band edges
ws = [0.4 0.6];
wp = [0.1 0.9];

%% Pass band and Stop Band attenuation
Ap = 3;
As = 18;

%% Get Cutoff and Order
[N wc] = buttord(wp,ws,Ap,As);
disp(['Order:' num2str(N)]);
disp(['Cutoff: ' num2str(wc)]);

```

```
%% Value of Cutoff in Analog equivalent
WW = tan(wc.*pi/2);
disp(['Cutoff in Analog: ' num2str(WW)]);

%% Get Filter Coefficients
[b a] = butter(N,wc,'stop')
%% Get Pole-Zero Plot
figure;
zplane(b,a);
set(findall(gcf,'Type','line'),'LineWidth',2,'MarkerSize',40)

%% Get Frequency Response
figure;
freqz(b,a);
set(findall(gcf,'Type','line'),'LineWidth',2,'MarkerSize',40)
```

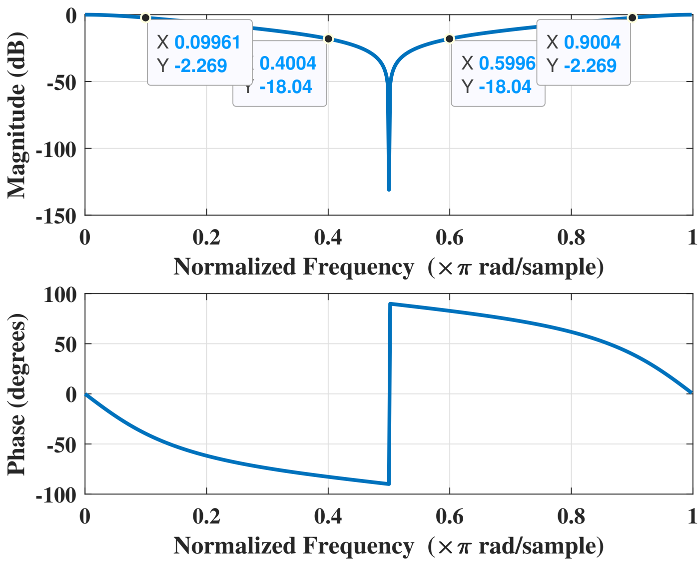

Figure 15: Frequency response for BSF

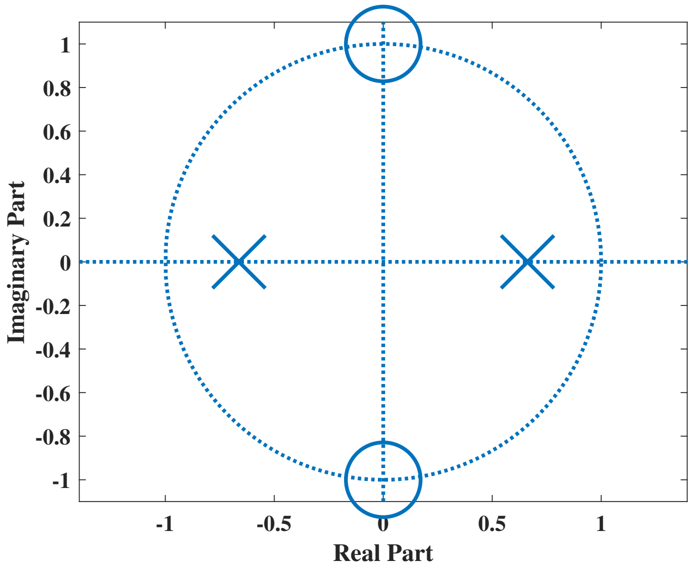

Figure 16: Pole zero for SBF

7 Summary:

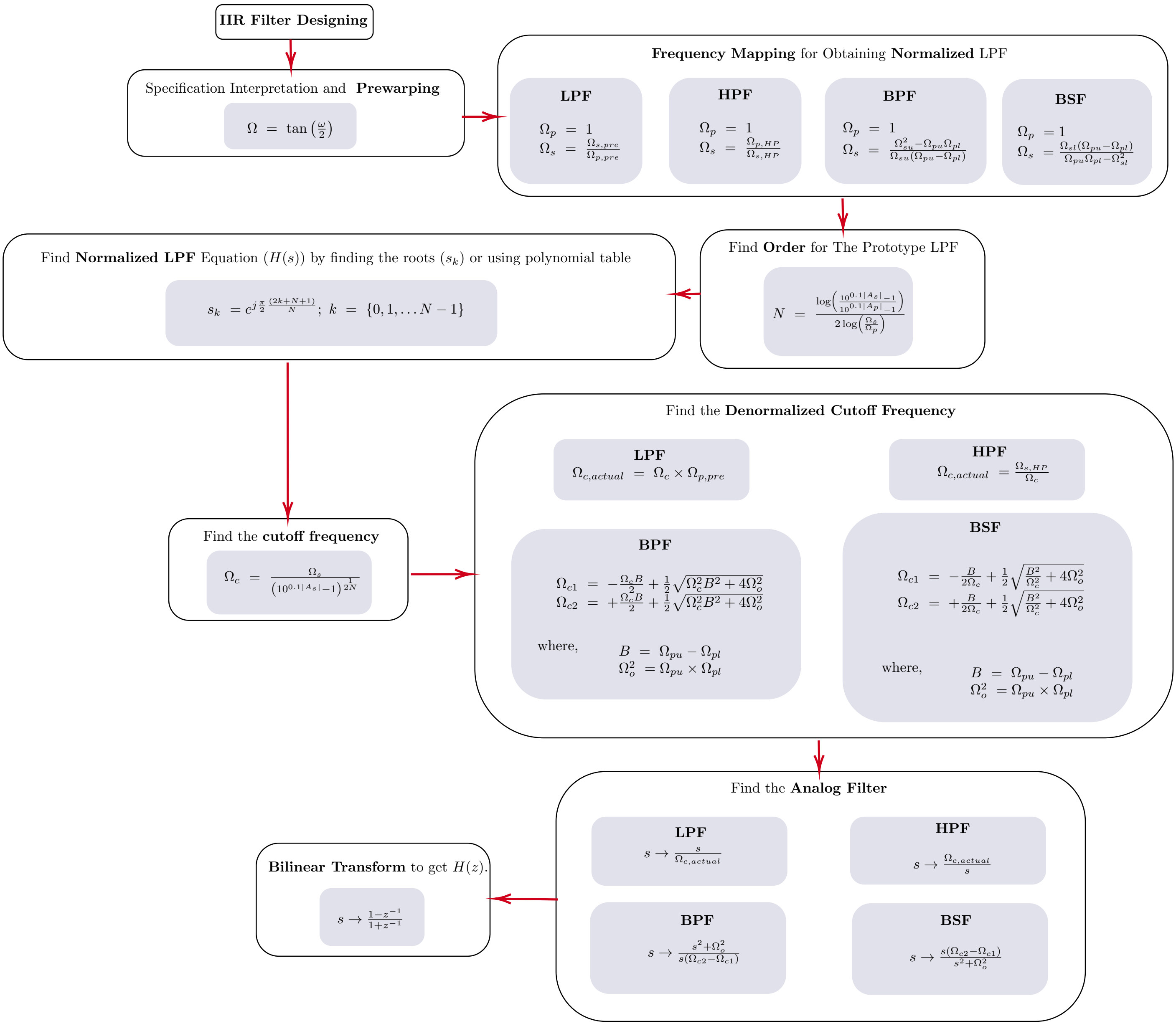

Figure 17: Summary of the IIR Filter Design

Table 2: Typical Frequency Mappings

Frequencies	**Frequencies Mapped for prototype LPF**			
	LPF	**HPF**	**BPF**	**BSF**
Ω	$\frac{\Omega}{\Omega_{p,LP}}$	$-\frac{\Omega_{p,HP}}{\Omega}$	$\frac{\Omega^2-\Omega_{pu}\times\Omega_{pl}}{\Omega(\Omega_{pu}-\Omega_{pl})}$	$\frac{\Omega(\Omega_{pu}-\Omega_{pl})}{\Omega_{pu}\times\Omega_{pl}-\Omega^2}$
$-\infty$	$-\infty$	0	$-\infty$	0
0	0	∞	∞	0
∞	∞	0	∞	0
Center Frequency (Ω_o) (For BPF and BSF)	-	-	0	∞
$-\Omega_p$	-1	1	-	-
Ω_p	1	-1	-	-
$-\Omega_s$	$-\frac{\Omega_s}{\Omega_{p,LP}}$	$\frac{\Omega_{p,HP}}{\Omega_s}$	-	-
Ω_s	$\frac{\Omega_s}{\Omega_{p,LP}}$	$\frac{\Omega_{p,HP}}{\Omega}$	-	-
Ω_{pl}	-	-	-1	1
Ω_{pu}	-	-	1	-1
Ω_{sl}	-	-	$\frac{\Omega_{sl}^2-\Omega_{pu}\times\Omega_{pl}}{\Omega_{sl}(\Omega_{pu}-\Omega_{pl})} = -\frac{\Omega_{su}^2-\Omega_{pu}\times\Omega_{pl}}{\Omega_{su}(\Omega_{pu}-\Omega_{pl})}$	$\frac{\Omega_{sl}(\Omega_{pu}-\Omega_{pl})}{\Omega_{pu}\times\Omega_{pl}-\Omega_{sl}^2}$
Ω_{su}	-	-	$\frac{\Omega_{su}^2-\Omega_{pu}\times\Omega_{pl}}{\Omega_{su}(\Omega_{pu}\Omega_{pl})}$	$\frac{\Omega_{su}(\Omega_{pu}-\Omega_{pl})}{\Omega_{pu}\times\Omega_{pl}-\Omega_{su}^2} = -\frac{\Omega_{sl}(\Omega_{pu}-\Omega_{pl})}{\Omega_{pu}\times\Omega_{pl}-\Omega_{sl}^2}$