Computational Social Choice **Winter 2017**

Lecture 1: Basic Concepts

Instructor: Prof. Edith Elkind *Scribes: Varad P.*

1.1 Rank Aggregation

Rank Aggregation also known as 'voting' is the problem of aggregating several ordered lists of alternatives,

Input:

- A set of alternatives (candidates): $C = \{C_1, C_2, ...C_m\}$
- A set of voters: $V = \{1, 2, 3, ..., n\}$
- For each voter, a total (ranking) order over C.

Output:

- A winner
- A set of winners
- A total ranking of alternatives.

1.2 Examples

1.2.1 Political Voting

Suppose there is an election in United Kingdoms.

The candidates are: Conservatives (C), Labour (L), Liberal Democrats (LD) Suppose there are 60,000 voters.

Now, the voters vote as follows:

25,000 voters: $C > LD > L$

20,000 voters: $L > LD > C$

11,000 voters: $LD > L > C$

4,000 voters: $LD > L > C$

If, for example Plurality voting method is used, we get C as a winner with 25,000 votes.

1.2.2 Competition for a Fellowship

No. of Candidates is 50

Voters: A panel of 15 members. Note that each panel has a ranking of candidates.

Goal: To select 10 candidates for a fellowship.

Standard Approach:

- Each voter is asked to identify top 10 candidates, and order them
- Each student gets s_i points from each voter who ranks him in position i, where $s_1 \geq ... \geq s_{10}$
- The students with top 10 total scores win.

1.2.3 Ranking of Universities

A panel is supposed to tank UK universities.

Ranking are based on 5 different criteria:

1. Reputation Ranking
2. Grant Income
3. Student Satisfaction
4. Number of research papers published
5. Average salary after graduation

Suppose, the ranking according to different criteria are as follows:

- criterion 1: Cambridge >Oxford >UCL >LSE
- criterion 2: Oxford >Cambridge >LSE >UCL
- criterion 3: UCL >Cambridge >Oxford >LSE
- criterion 4: Oxford >LSE >Cambridge >UCL
- criterion 5: LSE >Cambridge >UCL >Oxford

Should all criteria have the same weight? Which might be the most optimal method so that the rankings are made fairly.

1.3 Single Winner Rules

1.3.1 Plurality

- each voter names his favorite candidate
- candidates with the largest number of votes win
- if two or more candidates get the highest score, the winner is chosen using some tie-breaking rule

Plurality is obviously the best voting rule if there are 2 candidates.

However, for 3 candidates it may behave in an undesirable way. The Majority of the voters may prefer some other alternative to the current winner. Voters have an incentive to vote non-truthfully. We will discuss about this and manupulation in the next lecture.

Example: Refer to the Political Voting example. In it, the candidates are: Conservatives (C), Labour (L), Liberal Democrats (LD) Suppose there are 60,000 voters.

Now, the voters vote as follows:

25,000 voters: $C > LD > L$

20,000 voters: $L > LD > C$

11,000 voters: $LD > L > C$

4,000 voters: $LD > L > C$

Using Plurality, the outcome is C is the winner.
But, it can be seen that 31,000 voters prefer LD to C. Thus, the voters ranking $LD > L > C$ would be better off voting L.

1.3.2 Multi-Round Elections

- All voters vote for their favorite candidate.
- If some candidate gets more than 50% of the votes, he is declared the winner.
- Otherwise, the candidate with the smallest number of votes is eliminated
- The voters are asked to vote again over the remaining candidates
- The process repeats until some candidate gets a majority of votes

Example: Let us take the same example as that in Plurality, in the first round, C is the winner and LD has the least amount of votes, thus LD is eliminitated. Now, in the second round, L and C get 11,000 and 4,000 votes respectively. Thus, now L has the majority, it wins.

1.3.3 Single Transferable Vote

It is to be noted that Multi-Round elections often produce a more appealing outcome than Plurality. However, they are harder to implement. (Voters need to come to booth many time) This is something very time and money consuming.

So, an implementation of the multi-round method called Single Transferable Vote, or simply STV was proposed.
Single Transferable Vote: An implementation of multi-round elections in a single round of voting.

- each voter submits a total ranking of candidates
- the election authority simulates multi-winner elections based on the information in the ballots (assuming that all voters always vote for their most preferred available candidate)

How good is STV?

Refer to the Political Voting example. In it, the candidates are: Conservatives (C), Labour (L), Liberal Democrats (LD) Suppose there are 60,000 voters.

Now, the voters vote as follows:

25,000 voters: $C > LD > L$

20,000 voters: $L > LD > C$

11,000 voters: $LD > L > C$

4,000 voters: $LD > L > C$

Plurality chooses C, STV chooses L.
Yet, 40 000 voters prefer LD to L and 35 000 voters prefer LD to C. Under both Plurality and STV, more than 50% of voters would have preferred a different candidate Under STV, the voters who rank C first would be better off voting for LD

1.3.4 Condorcet Winners

Suppose each of the n voters have a ranking of all m candidates.

Definition: A candidate c wins a pairwise election against a candidate d if more than half of the voters rank c above d.

A candidate is said to be a Condorcet winner if he wins in all pairwise elections he participates in.

In the above example, 'a' is the Condorcet winner.

1.3.4.1 Condorcet Consistency

A voting rule is said to be Condorcet-consistent if it selects the Condorcet winner whenever it exists.

Refer to the Political Voting example. In it, the candidates are: Conservatives (C), Labour (L), Liberal Democrats (LD) Suppose there are 60,000 voters.

Now, the voters vote as follows:

$$\text{25,000 voters: } C > LD > L$$

$$\text{20,000 voters: } L > LD > C$$

$$\text{11,000 voters: } LD > L > C$$

$$\text{4,000 voters: } LD > L > C$$

Plurality chooses C, STV chooses L. But, LD is the Condorcet winner. LD is the condorcet winner even thought it has the smallest number of voters voting it first.

Thus, it can also be seen that neither Plurality nor Multi-Round method are Condorcet Consistent.

1.3.4.2 Does Condorcet-Consistency also stay true?

Suppose the voting takes place as follows:

$$A > B > C$$

$$B > C > A$$

$$C > A > B$$

Thus, it can be seen that there is no Condorcet winner! All candidates have the same number of pairwise wins against each other.

Definition: G is a pairwise majority graph for an election E with a candidate set C if its vertex set is C and there is an edge from a to b iff majority of voters prefer a to b.

Theorem: For any directed graph G = (V, E) on n vertices that does not have cycles of length 2, there is an election with the candidate set V and at most n(n-1) voters whose pairwise majority graph coincides with G

Proof of the Theorem: For each edge (x, z) of G, we create two voters

- v^1_{xz}: x >z >all other candidates in lex order.

- v_{xz}^2: all other candidates in reverse lex order >x >z

As there are n candidates, there will be n(n-1)/2 edges, meaning there are n(n-1) voters.

Fix two vertices x and z:

1. If there is no edge between x and z in G, x and z are tied.
   In each pair of voters, 1 voter ranks x above z, and 1 voter ranks z above x.
2. If (x,z) is in G, x beats z in their pairwise election
   In all pairs other than (v_{xz}^1, v_{xz}^2)
   In (v_{xz}^1, v_{xz}^2) both voters rank x above z

1.4 Condorcet-Consistent Rules

A Condorcet-consistent rule must elect a Condorcet winner when one exists

1.4.1 Copeland Rule

Copeland Rule states that each candidate gets:

- 1 point for each pairwise election he wins
- 0.5 points for each pairwise election he ties

The candidate with the largest number of points wins.

Thus, in a m-candidate election, a Condorcet Winner exists if he gets m–1 points and the other candidates get atmost m-2 points.

1.4.2 Maximin Rule

Maximin rule states that the score of each candidate is the number of votes he gets in his worst pairwise election, the candidate with the highest number of votes wins.

In an n-voter election, if a Condorcet-Winner exists:

- His maximin score is greater than n/2
- Everyone else's Maximin score is less then n/2

1.5 Scoring Rules

Condorcet-Conssitent rules are hard to explain to voters (as implementation is non-trivial)

<u>Alternative:</u> Scoring Rules

A scoring rule for an election with m candidates is given by a vector $(s_1, ..., s_m)$, where

$$s_1 \geq ... \geq s_m$$

Each candidate gets s_i points from each voter who ranks him i^{th}
Candidate with the maximum number of points wins.

Rules:

- Borda: (m-1, m-2, ..., 2, 1, 0)
- k-approval: (1, ..., 1, 0, ..., 0), where k means the no. of candidates a voter can vote for.

1.5.1 Example

Refer to the Political Voting example. In it, the candidates are: Conservatives (C), Labour (L), Liberal Democrats (LD) Suppose there are 60,000 voters.

Now, the voters vote as follows:

25,000 voters: $C > LD > L$

20,000 voters: $L > LD > C$

11,000 voters: $LD > L > C$

4,000 voters: $LD > L > C$

Pluraity: C wins with 25,000 points

Borda: LD wins with 75,000 points

- C gets 2 x 25 000 + 1 x 4 000 = 54,000 points
- L gets 2 x 20 000 + 1 x 11 000 = 51,000 points
- LD gets 1 x 45 000 + 2 x 15 000 = 75,000 points

2-Approval: LD wins with 60,000 points

- C gets 25 000 + 4 000 = 29,000 points
- L gets 20 000 + 11 000 = 31,000 points
- LD gets 45 000 + 15 000 = 60,000 points

1.5.2 Pros and Cons

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Lecture 2: Likelihoods and Distances

Instructor: Prof. Edith Elkind *Scribes: Varad P.*

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Lecture 3: Strategic Behavior

Instructor: Prof. Edith Elkind *Scribes: Varad P.*

When voters start voting strategically, it leads to some very interesting results. If someone wants to try change the results of an election, it can be done in the following ways: "Bribery", "Campaign Management", "Control" and "Strategic Voting". In this lecture, we will only discuss about strategic voting, also called "Manipulation".

3.1 Manipulation

3.1.1 Plurality

Consider candidates a, b, c. Suppose that under the truthful voting, a gets 20 votes, b gets 20 votes and c gets 5 votes. Ties are broken lexicographically. That means, that votes for c are wasted, as c would never win. As ties are broken lexicographically, 'a' wins.

Manipulation: to let b win, a voter who is voting in the order c >b >a, would instead vote b >c >a. Thus, now 'b' wins.

3.1.2 STV

Consider candidates a, b, c, d.

Let there be three voters such that:

$$a > b > c > d$$

$$c > b > d > a$$

$$d > b > c > a$$

If we use the STV in the above election, it can be seen that 'c' is the winner.

Manipulation: Suppose, the first voter is a guy who believes in a Condorcet Winner, and thus, according to him, 'b' should be the winner. Thus, what he can do is that he can vote:

$$b > a > d > c$$

Now, it can be seen that 'b' wins.

3.2 Gibbard-Satterthwaite Theorem

Suppose the no of Candidates $C \leq 3$. Then for any voting rule R such that R is not a 'dictatorship' and Im(R) = C there exists a list of voters preferences such that some voter v has an incentive to 'manipulate (vote non-truthfully)'

That means that v can change his vote so that the winner is a candidate that v ranks higher than the original winner.

3.3 Complexity as a Barrier to Manipulation

This was proposed by [**Bartholdi, Tovey, Trick '89**]. Suppose, we know all the other voters and also the voting rules, Then, the question is whether we can figure out if we have a **successful manipulation?**

The answer is sort of obvious for Plurality. But, what about other rules? The answer is non-trivial for STV, and same is the case for other voting rules. Manipulation is polynomial time for some voting rules and therefore easy, and it is NP–Complete for some voting rules and therefore hard.

Hope: If manipulations are to find in the worst case, it might be possible that they are not an issue in practise?

3.3.1 Computational Setup

Let us define manipulation formally so as to reason about its hardness. Suppose there is a rule F. Then, for a Manipulation(F):

Input: Ballot of voters 1, ..., n-1, candidate c.

Question: Can voter n vote so as to make 'c' win under rule F? So, there are two possibilities of c winning. This leads to:

- Unique Winner Problem(UniqueW)
  Can voter n make c the unique winner?
- Co-Winner Problem(CoW)
  Can voter n vote so as that c is tied for winning?

Note 1. It is interesting to note that the complexity of both the cases is the same. That is, if one is NP–Hard, the other one is also NP–Hard.

If Manipulation(F) is easy, finding an improving manipulation is easy. Now, how is this related to the Gibbard-Satterthwaite Theorem? If we figure out the above question, then we can determine if we have an improving manipulation. Suppose there is a candidate W, look for every candidate ranked above W, if any one of them shows that manipulation is possible, then we say that we have an improving manipulation. We can make anyone we prefer to the current winner to win the election.

Note 2. The manipulation problem is easy if the number of candidates is small.
Example. if we have 3 candidates, we have 3! votes to consider.

3.3.2 A Meta Algorithm (CoW)

Let F be the Scoring Rule. Let us assume that we, as the manipulator know other voters preferences. It is obvious that the manipualative vote is as follows:

- Rank preferred candidate c in the first position.
- Total score of c can be calculated.
- If score of c is not the highest

  return FAIL
- Else, for i = 2, ..., m

  Check if some candidate can be safely placed in next position.

  If so, place it there.
  Else, FAIL.
- If true for all candidates, we have a Manipulative vote!

The above algorithm is **Polynomial**. It is to be noted that the above algorithm is not only valid for the scoring rule but also for other score rules.

Exercise 1. Prove the Meta Algorithm (CoW).

3.3.2.1 Applicability

The Meta Algorithm (CoW) works for Copeland Rule also.

Mechanism:

- Place preferred candidate c in the first position.
- Total Copeland score of 'c' can be calculated.
- Same as before, once we place candidate in position i, with the first i–1 positions filled, we know its Copeland Score.

The same can be said for **Maximin**. In short, the algorithm is said to be true for rules based on scores.

Think a bit: Doesn't that mean that rules like Borda, Copeland, Maximin can be easily manipulated in polynomial time ??

Note 3. The Meta Algorithm was made for the Co-Winner Problem, but, it can also be modified very easily for the Unique Winner Problem.

3.3.3 Hope against Manipulation?

As we saw in the previous section, manipulation of even the fairly complex voting rules is easy. Next, we see the Single Transverable Vote (STV). STV a voting rule, is not based on scores. In fact, manipulation under STV is NP–Hard []**Bartholdi, Orlin '91**]. Even under lexicographical tie-breaking. Which is a good news for us, because this means that we finally have some protection against manipulation to some extent.

The next question naturally arises. Is STV resisitant in manipulation in real life? Unfortunately, on real-life preference data, manipulation seems to be easy to find.

Think a bit: The reason for this to happen, might be because we are only taking some instances of STV in real life while here we are cansidering all the cases includeing the worst cases beacuse of which it is NP–Hard.

3.3.4 Coalitional Manipulation

Arguably, manipulation by a single voter is often not an issue. No single voter is pivotal. So, why not intriduce a bunch of manipulators.

3.3.4.1 Setup

Let there be a group of M of k voters and a candidate c.
We have the ballot of the remaining n–k voters with us.

Question: Can voters in M vote so as to make c a winner under F?
Similar, to earlier we have two types of winners:

- Unique Winner Problem
- Co-Winner Problem

3.3.4.2 Rules for which Manipulation is Easy

Let's start studying Coalitional Manipulation starting with the easiest voting rule, 'Plurality'. It is pretty obvious that all the manipulators in group M should vote for the candidate c.

The next easy voting rule is 'k-approval'. Suppose, k = 2. Then:

- Each manipulator should rank c and some other candidate in the top two positions.
- Suppose, before manipulation, a candidate x gets s_x points.
- Then, atmost $s_c - s_x + k$ manipulators a=can vote for the candidate x.

3.3.4.3 Rules for which Manipulation is Hard

Theorem: Coalitional Manipulation is NP–Hard for the fairly complicated rules like Copeland and Maximin. It is also NP–Hard for scoring rules like Borda.

Proof: Proof for the Borda rule will be discussed in the next Tutorial.

Note 4. The rules are NP–Hard even for just 2 manipulators. As STV was NP–Hard even for just 1 manipulator, it will obviously be NP–Hard for coalitional manipulators.

3.3.5 Weighted Coalitional Manipulation

Suppose, there is a company and an election is being held for electing a Chairman. Now, there are some candidates contesting for that position, and there are a group of manipulators who want to elect a specific candidate. So, the manipulation will depend on the weight of the voter who is manipulating, more is his weight, more will be the manipulation he can do.

For single voter, it doesn't make much of a difference. The Meta Algo. works fine.

Suppose voter i has weight W_i
Manipulation in weighted coalition will be hard. As always, Plurality will be easy. For other rules, successful manipulation might have to split M into groups of equal weights. It is atleast as hard as Partition Problem. Thus, it is hard even for M=3.

3.3.5.1 Hardness Proof

Theorem: Weighted Coalitional Manipulation is NP-Complete under Borda even for 3 candidates.

<u>**Construction**</u>

Partition Problem: Let there be t numbers $a_1, a_2..., a_t$ such that

$$\sum a_i = 2K$$

quest. Is there a subset I with $\sum_{i \in I} a_i = K$?

Candidates: a, b, p. 'p' being the preferred candidate.

<u>Proof</u>

Let us assume that in a pool of voters, there are only two truthful voters and the rest are the manipulators trying to make the 'p' candidate win. If the two voters vote as follows:

$$1 \text{ voter: } a > b > p, \text{weight 3K}$$

$$1 \text{ voter: } b > a > p, \text{weight 3K}$$

Therefore, according to Borda Rule, the scores will be: 'a' and 'b' score 9K each and 'p' scores 0. Now, let there be t manipulators with weights $3a_1, ..., 3a_t$. It is pretty obvious that each manipulator will rank 'p' first. Therefore, p's score will be (weight X score per voter X no of voters) which is 12K. Pretty good, we beat both a and b. But, the problem is that either a or b will have their score increased by 6K. We don't want that. So, we can split the voters into two equal parts and make them vote as follows:

$$\text{K voters: } p > b > a, \text{weight 3}$$

$$\text{K voters: } p > a > b, \text{weight 3}$$

Now, the scores will be: a = b = p = 15K. So, our preferred candidate is a co-winner.

This means that if we are not able to split the votes evenly in two groups, then we will always get a candidate whose votes are more than 15K and that means, we won't be able to make our preferred candidate a winner. That means, that Partition Problem needs to work if we want even spilt of votes. That means that our manipulation problem is atleast as hard as Partition Problem. That means, that Weighted Coalition Manipulation is NP–Hard.

3.3.6 Destructive Manupulation

Till now we have seen what will happen if we want to make preferred candidate win. Now, suppose we don't want a candidate to win, what should we do? Make sure that there is always someone ahead of him.

<u>Observation:</u> It can be seen that if Constructive Manipulation (CoW) is easy, then Destructive Manipulation (UniqueW) is also easy. But, it should be noted that the converse is not true.

This can be seen with an example. In veto rule, the scoring rule is a vector {1, 1, 1, ..., 1, 0} Suppose, we take weighted veto for 3 candidates, now we have to make sure that our preferred candidate is atleast the cowinner in the set of three candidates. And the manipulation for this construction is hard. Thus, **converse is not true.** Constructive Plurality is similar to Destructive Veto. Therefore, destructive manipulation is easy.

3.4 Randomized Tie-Breaking

Till now, we have studied Unique Winner and Co-Winner Problems. In Co-Winner problems, we have assumed that our preferred candidate will always win in the tie-break. So, the most natural question to ask is that suppose a winner is choosen among tied candidates uniformly at random, what is a successful manipulation?

Suppose in plurality, candidates are given the following votes:
a: 5 points, b: 5 points, c: 4 points
Now, if the manipulator's preference is $c > b > a$, it can be a three way tie, which can be broke randomly with each having equal chances of winning (if voted truthfully). If we don't like 'a', we can give the vote to 'b' and make it the unique winner. Therefore, we can't actually say what is a successful manipulation if the only information with us is the preference order!

Now, let us assume that manipulator has utilities.

- if u(c) = 10, u(b) = 2 and u(a) = 0, average utility = $\frac{\sum_{i=1}^{3} u(i)}{3} = 4$. Therefore, the manipulator will select {a, b, c}
- if u(c) = 10, u(b) = 8 and u(a) = 0, average utility = $\frac{\sum_{i=1}^{3} u(i)}{3} = 6$. Therefore, the manipulator will select {b}

3.4.1 Complexity of Manipulation

Let us study the complexity witht the help of some voting rules. Let us start with Plurality:

T: Set of candidates with the highest score (s)
N: Set of candidates with the score (s–1)

Now, the manipulator can do two things:

- First, he can choose a preferred candidate from the set T and make the unique winner of the election, or
- He can choose his preferred candidate from the set N and make him the co-winner with the candidates in set T.

Theorem: Manipulation under randomized tie-breaking is in polynomial time for all scoring rules like Borda. Interestingly, it is NP-Complete for rules like Copeland and Maximin.

3.5 Voting Equilibria

Manipulation Setting: The manipulator assumes that she is the only strategic voter. That means, the manipulator assumes that everyone else votes truthfully.

The next question that comes to mind is what will happen if all the voters are strategic players? Then, it will be the case of game-theoretic solution concept. The most classic one is **'Nash Equilibria (NE)'**. Nash Equilibrium is a collection of actions, one per player, so that no one wants to change the action given other players action. So, in the context of voting, you don't want to change your vote given how everyone else votes.

3.5.1 Nash Equilibria in voting

Typically, NE is studied for plurality rule as it complex even for plurality.

3.5.1.1 Bad Equilibrium Alert

Let us suppose that all voters agree to vote in the same preference order as follows:

$$A > B ... > Z$$

Therefore, everyone voting for Z is a Nash Equilibrium. That means even if a manipulator votes for A, the plurality outcome will be Z only. That means, this is a very bad outcome. So, in such situations, no one can unilaterally change the situation.

3.5.1.2 Lazy Voters

Let us suppose that all voters agree to vote in the same preference order as follows:

$$A > B ... > Z$$

Lazy Voters: Suppose each voter is lazy, meaning he prefers not to voter (Φ) if his vote is changes nothing. Now, (Z,Z, ..., Z) is no longer a Nash Equilibrium. But, $(A, \Phi, ..., \Phi)$ is a Nash Equilibria! This is beacuse, only the first voter marks his preference, and the rest of the voters are in equlibria, hence, the answer,

3.5.1.3 Truth-Biased Voters

Let us suppose that all voters agree to vote in the same preference order as follows:

$$A > B ... > Z$$

Truth-biased Voters: Now, suppose each voter prefers to vote truthfully if his vote changes nothing. Now, (Z,Z, ..., Z) is no longer a Nash Equilibrium. But, $(A, A, ..., A)$ is a Nash Equilibria!

3.5.2 Characterization of Randomized Tie-Breaking

Nash equilibria with unique winners:

- (X, X, ..., X) is not a NE; then what about $(\Phi, \Phi, ..., , \Phi, X)$?
- If voter 1 prefers Y to X, they can change the outcome to $\{X,Y\} >_1 X$
- X is a unique winner (with one vote) if all the voters rank X first.

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Lecture 4: Restricted preferences

Instructor: Prof. Edith Elkind — *Scribes: Varad P.*

Lecture Summary:

1. preferences are completely arbitary.
2. If voters preferences are structured:
   - Impossibility results might disappear
   - Algorithmic complexity results might also disappear
3. Examples of Strucutred preferences of this case:
   - Single Peakedness preferences
   - Single Crossed preferences, etc.

4.1 Single Peaked preferences

Definition: A preference profile is single-peaked (SP) with respect to an ordering < of candidates (axis) if for each voter v:

- if top(v) < D < E, v prefers D to E.
- if A < B < top(v), v prefers B to A.

Example: Suppose, there are 6 candidates standing for an election, and the following preference profile is given. Then,

$$C > B > D > E > F > A$$
$$A > B > C > D > E > F$$
$$E > F > D > C > B > A$$

As we can see in the above examples, the geometric shape of the preferences give rise to a single peak. Thus, the name 'Single-Peakedness'.

4.2 Example

Refer to the Political Voting example. In it, the candidates are: Conservatives (C), Labour (L), Liberal Democrats (LD) Suppose there are 60,000 voters.

Now, the voters vote as follows:

$$25{,}000 \text{ voters: } C > LD > L$$

$$20{,}000 \text{ voters: } L > LD > C$$

$$11{,}000 \text{ voters: } LD > L > C$$

$$4{,}000 \text{ voters: } LD > C > L$$

Another example is of the water temperature at a beach. Some like cold water while some like hot water. But, most of them will prefer something in between. That is to say, it will be a downward incline towards both the extreme points.

In case of room temperature though, it is a bit complicated as there are a lot of variations in that case.

4.3 SP preferences

4.3.1 Transitivity

In a condorcet cycle, majority prefers A to B, B to C and C to A. That is, it is not a transitive closure. In **weakly** majority relation, atleast half of the voters prefer A to B, so it can exactly half. But, we cannot always hope for majority preference relation to remain transitive. Majority preference relation consists of ties, while the **weakly** majority preference relation can be said to be **weakly** transitive.

Theorem: In Single-Peaked elections, the (weak) majority relation is transitive.

Lemma: There exists a candidate preferred to every other candidate by a (weak) majority of voters (The Condorcet Winner).

Solution by using Induction:

1. Assuming that the lemma is true, let us say that there is a Condorcet Winner 'a'.
2. Now, delete 'a' from all the voters, the profile remains SP.
3. Similarly, take the next candidate which was weakly losing to 'a'. Repeat the same for it.
4. In this way, we will be able to get a preference order.

Steps:

- order the voters according to their top choice.
- if we have n = 2k + 1 voters, top(v_{k+1}) is a CW
- if we have n = 2k voters, all candidates between top(v_k) and top(v_{k+1}) are weak CWs.

When there are odd number of voters, it is easy as there will be always a single condorcet winner. But, in case of even number of voters, there might be ties. Thus, it leads to weakly condorcet winners.

4.3.2 Circumventing Gibbard-Satterthwaite

Suppose we have an odd number of voters. That is, n = 2k + 1. Then, median voter rule says the following:

> Consider an election that is single-peakedness with respect to '<'. Ask each voter to vote for a candidate and let C(v) denote the vote of voter v. Order voters according to C(v) and break the ties arbitarily.
> Finally, output $C^* = C(v_{k+1})$

Claim: This voting rule is not manipulable. The best choice for the voters is to vote truthfully.

Theorem: Under the median rule, it is a dominant strategy to vote for one's top choice.

Interpretation: Consider a voter v_i in the pool of voters.

Case 1: i = k+1. v_i gets his most preferred outcome.

Case 2: i < k+1 or i > k+1. If v_i votes for a candidate C, c $\leq C^*$, v_{k+1} remains the median voter, so the outcome does not change.

Case 3: If v_i votes for C, C > C^*, either v_i (with his new vote) or v_{k+2} becomes the median voter, so the outcome becomes worse for v_i.

4.3.3 Equivalent Definitions

4.3.3.1 Definitions

Our Definition (Decreasing from the peak): A vote v is SP with respect to an ordering '<' if:

(1) if top(v) < D < E, v prefers D to E.

(2) if A < B < top(v), v prefers B to A.

Alternate Definition (No Valley): A vote is SP with respect to an ordering '<' if for every triple candidates A < B < C, it holds that B is not the voter's least preferred candidate in {A, B, C}

Alternate Definition (Continuous Segments): A vote is SP with respect to an ordering < if for every k, the set of top k candidates in v forms a continuous segment with respect to <

4.3.3.2 Proofs

Decreasing from the peak to no valleys: Suppose for contradiction v ranks B last of {A, B, C}. Suppose top(v) is to the left of B. Then we have top(v) < b < C, so v must prefer B to C. Suppose top(v) is to the right of B, then we have A < B < top(v), so v must prefer B to A. That means there are no valleys.

No valleys to Contiguous Segments: Suppose for the contradiction, the set in the top k candidates in v isnot contiguous. Then, pick the smalleset such k. Let D be v's k^{th} candidate. Let Z be the candidate that seperates D from the top k–1 candidates in v. Assume top(v) > Z < D. As v ranks Z below D, {top(v), Z, D} forms a valley.

Continuous Segments to Decreasing from the peak: Suppose for contradiction, top(v) < D < C, yet v prefers C to D. Now, suppose v ranks C in position k then, v ranks D in k+1 position or even lower than that. Thus, the set of top k candidates in v is not contiguous with respect to <

4.3.4 Properties

Claim: If a profile is single-peaked, then the set of candidates that are ranked last by atleast one voter has size ≤. Only the endpoints of the axis can be ranked last.

Claim: A profile over m candidates can be single peaked with respect to $\leq 2^{m-1}$ different axes, and this bound is tight.

Claim: If there are 2 voteres only in a single-peaked profile, and they vote as follows: $A < B < ... < Z$ and $A > B > ... > Z$, then the only option for the candidates will to have arrange the candidates in the asending or descending orders, and A and Z will always will gave to be the endpoints.

4.4 Single-Crossing Preferences

Definition: A profile is single-crossing (SC) with respect to an ordering of voters $(v_1, v_2, ..., v_n)$, if for each pair of voters A and B, there exists an i ∈ {0, 1, ..., n} such that

- voters $(v_1, v_2, ..., v_i)$ prefer A to B, and
- the voters $(v_{i+1}, v_{i+2}, ..., v_n)$ prefer B to A.

Example:

' A	B	B	C	C	C	D
B	A	C	B	B	D	C
C	C	A	A	D	B	B
D	D	D	D	A	A	A

Table 4.1: Voters Preferences

4.5 SC Preferences

4.5.1 Majority is Transitivity

Claim: In single-crossing elections, the majority relation is (weakly) transitive. For an odd number of voters, that is, n = 2k+1, if we that the v_{k+1}